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Three-loop calculation of the random random walk problem: an application of dimensional transformation and the uniqueness method

S É Derkachov†, J Honkonen‡ and Yu M Pis'mak†

† Department of Theoretical Physics, State University of Leningrad, Ul'yanovskaya 1, Staryi Petergof, 198904 Leningrad, USSR

‡ Research Institute for Theoretical Physics, University of Helsinki, Siltavuorenpenger 20 C, SF-00170 Helsinki, Finland

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Abstract. The method and details of the calculation of the three-loop contribution to the anomalous dimension of the diffusion coefficient of the model of a random walk in a potential random field with long-range correlations are presented. Contrary to earlier conjectures, this contribution does not vanish identically. A new method of calculation of multi-loop Feynman graphs with complicated numerator structure is suggested. It leads to simpler integrals in a space of higher dimensionality, which are computed using the recursion relations of the uniqueness method.

1. Introduction

We consider the problem of diffusion in a potential random velocity field described by the equation

$$[-\mathrm{i}\omega - D_0\nabla(\nabla\psi(\boldsymbol{x}) + \nabla)]c(\boldsymbol{x},\omega) = 0 \tag{1}$$

where c is the density (Fourier-transformed over the time variable) of diffusing particles, D_0 is the bare (not renormalized) diffusion coefficient, and ψ is the random potential with zero mean and the correlation function defined by the Fourier transformation F of the (generalized, when necessary) function $1/|\boldsymbol{p}|^{2+2\alpha}$: $C_0(\boldsymbol{x}-\boldsymbol{x}') \equiv \langle\psi(\boldsymbol{x})\psi(\boldsymbol{x}')\rangle_0 = g_0(2\pi)^{-d}F[1/|\boldsymbol{p}|^{2+2\alpha}](\boldsymbol{x}-\boldsymbol{x}')$, i.e.

$$C_0(\boldsymbol{x}-\boldsymbol{x}') = \begin{cases} \dfrac{2g_0 \ln|\boldsymbol{x}-\boldsymbol{x}'|}{(4\pi)^{1+\alpha}\Gamma(1+\alpha)} & \text{if } d = 2+2\alpha \\ \dfrac{g_0\Gamma(d/2-1-\alpha)}{(4\pi)^{d/2}\Gamma(1+\alpha)}\left(\dfrac{2}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)^{d-2-2\alpha} & \text{otherwise.} \end{cases} \tag{2}$$

Here, Γ is the gamma function and the (non-negative) bare coupling constant g_0 describes the strength of the disorder. We have omitted the finite additive constant, which may be present in the relations (2) for $d \leq 2+2\alpha$. Since the potential ψ enters equation (1) with a derivative, this constant is irrelevant.

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This is a special case of the problem of diffusion in a random field, which recently has been extensively studied [1–3]. The problem may be cast in a field-theoretic form [2], in which the asymptotic behaviour of the diffusion can be studied by the renormalization group (RG). In the present case with a potential random field, the renormalization-group beta function, which governs the long-time asymptotic behaviour of the model, turns out to be trivial [3–5] (i.e. all the loop contributions to it vanish) leading to disorder-dependent asymptotic behaviour at the upper critical dimension $d = d_c$, and strong-disorder regime below it $d < d_c$. Moreover, in one- and two-dimensional cases with logarithmically growing correlations of the random potential, two-loop and higher order contributions to the anomalous dimension vanish [3–7], i.e. it can be calculated perturbatively exactly. It has been shown recently by explicit three-loop calculation [7] that this is not true for arbitrary dimensions of space, contrary to earlier conjectures [3, 5, 6]. In this paper, we present the method and details of the calculation used to obtain this result, which were omitted in the earlier short account of this work [7]. We have used a few novel ideas in the calculation of multi-loop Feynman graphs where the structure of the numerator of the integrand is complicated. First, we have found it convenient to express traces of products of effective longitudinal vector propagators in terms of Gram determinants. Second, we have used an important formula, which allows us to remove the Gram determinants from the resulting integrals with a simultaneous shift by two in the dimensionality of space. We think that these tricks might be useful also in other problems than the present one, and therefore deserve detailed discussion.

The paper is organized as follows. in section 2 we construct a convenient, although not quite standard, graphical expression for the averaged Green function of the stochastic problem, equations (1) and (2). In section 3 the main tools of calculation are presented with the details of the calculation of the graphs, and section 4 is devoted to discussion of the results. In appendix 1 we give the proof of a fundamental lemma of dimensional transformation, and in appendix 2 the properties of a special two-loop graph are analysed.

2. Graphical expression for the Green function

We shall calculate the retarded Green function of equation (1) averaged over the random potential ψ. Using functional representation for the ψ average and the Green function G_ω of equation (1), we arrive at a field theory, whose renormalized action is of the form

$$S_R = -\frac{1}{2M^\epsilon}\psi C^{-1}\psi + \tilde{\varphi}[m + Z\nabla(\nabla + \nabla\psi)]\varphi \tag{3}$$

with the convention that all closed loops of bare $\varphi\tilde{\varphi}$ propagators are zero. All the necessary sums and integrals are implied in expression (3), and subsequent similar formulae. It has been shown [4] that the field theory (3) is multiplicatively renormalizable, and, moreover, it can be renormalized by a single renormalization constant Z. We have introduced the scaling parameter M in (3) and denoted by C the 'renormalized' correlation function C, which is obtained from the bare one C_0 by the substitution $g_0 \to g$. The renormalized 'mass' m is defined as $m = \mathrm{i}\omega Z/D_0$, and the parameter $\varepsilon = 2+2\alpha-d$, where d is the space dimensionality. The renormalization constant Z determines the anomalous dimension γ_D of the diffusion coefficient $\gamma_D = -M\partial \ln Z/\partial M|_0$, where the subscript indicates that the partial derivative is taken with fixed values of the bare parameters. Since the beta function of the model (3) is trivial, at the upper critical dimension the renormalized coupling constant g remains a free parameter, on which the anomalous dimension γ_D depends, and we choose $g = g_0$. The full propagator G of the renormalized field theory (3) is connected with the averaged Green function G_ω of equation (1) as follows $\langle G_\omega(\boldsymbol{x}, \boldsymbol{x}')\rangle_0 = ZD_0^{-1}G(\boldsymbol{x}, \boldsymbol{x}')$.

The anomalous dimension does not depend on the renormalized mass m of the model [9], therefore we calculate it in the massless theory, and henceforth set $m = 0$. To avoid infrared difficulties, we introduce the scaling parameter M as the infrared cutoff in the regularized $\psi\psi$ correlation function:

$$C_{\text{reg}}(\boldsymbol{x} - \boldsymbol{x}') = g \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^d} \frac{\exp[\mathrm{i}\boldsymbol{p}(\boldsymbol{x} - \boldsymbol{x}')]}{(\boldsymbol{p}^2 + M^2)^{1+\alpha}} . \tag{4}$$

To remove large-momentum divergences, we use dimensional regularization of the field theory (3) with the parameter $\varepsilon = 2+2\alpha-d$. The full propagator G of the renormalized massless field theory (3) may be found by averaging the solution $G_\psi(\boldsymbol{x}, \boldsymbol{y})$ of the stationary equation

$$Z\nabla[\nabla + \nabla\psi(\boldsymbol{x})]G_\psi(\boldsymbol{x}, \boldsymbol{y}) = -\delta(\boldsymbol{x} - \boldsymbol{y}) \tag{5}$$

over the random potential ψ with the weight $\exp(\psi C^{-1}\psi/2M^\varepsilon)$, i.e. $G(\boldsymbol{x} - \boldsymbol{y}) = \langle G_\psi(\boldsymbol{x}, \boldsymbol{y})\rangle$. Introducing the function $R(\boldsymbol{x}, \boldsymbol{y}; \psi) = Z_R^{-1} Z \exp[\psi(\boldsymbol{x})]G_\psi(\boldsymbol{x}, \boldsymbol{y})$, where Z_R is a new renormalization constant, and the matrix $\mathbf{T}$ (we use the same notation for functions and their Fourier transforms)

$$R(\boldsymbol{p}, \boldsymbol{q}; \psi) \equiv \frac{p_m}{p^2} T_{mn}(\boldsymbol{p}, \boldsymbol{q}) \frac{q_n}{q^2} \tag{6}$$

we obtain from (5) the expression [7]

$$\mathbf{T} = -Z_R^{-1}\left[\mathbf{S} + 2(1 + U\mathbf{S})^{-1} P^{\|}\right] . \tag{7}$$

Here, $\mathbf{S}$ is the matrix $S_{mn}(\boldsymbol{p}) \equiv \delta_{mn} - 2p_m p_n/p^2$ (note that $\mathbf{S}^2 = \mathbf{1}$), $P^{\|}$ is the longitudinal projection operator $P^{\|}_{mn}(\boldsymbol{p}) = p_m p_n/p^2$, and the function U is defined as $U(\boldsymbol{x}; \psi) \equiv \tanh[\psi(\boldsymbol{x})/2]$. The renormalization constant Z of the field theory (3) may be determined from the relation

$$Z = Z_R \exp(\tfrac{1}{2} C_{\text{reg}}(0))$$

up to finite renormalization [7].

Averaging expression (7) over the renormalized distribution of the random potential ψ we obtain a graphical representation of the function $\langle\mathbf{T}\rangle$, constructed from vector lines corresponding to $\mathbf{S}$, scalar lines corresponding to the $\psi\psi$ correlator (4) and vertices with two S-legs and any *odd* number of ψ-legs generated by the function $U = \tanh(\psi/2)$. The advantage of this representation compared with the original field theory (3) is that the number of graphs to be calculated is significantly reduced. However, the graphical expression for $\langle\mathbf{T}\rangle$ does not correspond to any multiplicatively renormalizable field theory, and to carry out the renormalization, we use the fact that the function $\langle R \rangle$ can be made finite by a suitable choice of the renormalization constant Z_R [7].

We define the self-energy matrix $\hat{\Sigma}$ as

$$\langle (1 + U\mathbf{S})^{-1} \rangle \equiv (1 - \hat{\Sigma}\,\mathbf{S})^{-1} \tag{8}$$

and introduce the function Q: $\langle R(p, q; \psi) \rangle \equiv (2\pi)^d \delta(p + q) Q(p)/p^2$ for which we obtain from (6), (7) and (8) the equation

$$Q(p) = Z_R^{-1} \left\{ 1 - 2\mathrm{Tr}\left[(1 - \hat{\Sigma}(p)\mathbf{S}(p))^{-1} P^{\|}(p) \right] \right\}.$$

We determine the renormalization constant Z_R by the singular in ε contributions only, and expanding Z_R and $\hat{\Sigma}$ in g

$$Z_R = 1 + Z_1 + Z_2 + Z_3 + \cdots \qquad \hat{\Sigma} = \hat{\Sigma}_1 + \hat{\Sigma}_2 + \hat{\Sigma}_3 + \cdots$$

and choosing the Z_i to ensure the absence of divergences in ε in the function Q to third order in g, we obtain

$$\begin{aligned} Z_1 &= -2\overline{\Sigma}_1 \\ Z_2 &= 2\overline{\Sigma}_1^2 - 2\overline{\Sigma}_2 \\ Z_3 &= -\tfrac{4}{3}\overline{\Sigma}_1^3 + 4\overline{\Sigma}_1\overline{\Sigma}_2 - 2\overline{(\Sigma_3 + \tfrac{1}{3}\Sigma_1^3)}\,. \end{aligned}$$

The extraction of the singular in ε part of a quantity is denoted by a bar above it. The constants Σ_i are defined as follows:

$$\hat{\Sigma}_{i,mn}(p) = \delta_{mn}[\Sigma_i + F_i(p^2)] + p_m p_n J_i(p^2)$$

where $F_i \to 0$ and $J_i < \infty$ in the limit $p \to 0$.

Graphical expressions for the matrices $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$ are shown in figure 1, and the three-loop graphs, which give rise to $\hat{\Sigma}_3$, in figure 2. The double lines stand for the matrix $\mathbf{S}$, and the full lines denote the regularized correlation function C_{reg}; the vertex factors are equal to unity.

3. Calculation of the three-loop graphs

To extract the constants Σ_i from the graphs of figure 1 and figure 2, it is sufficient to calculate them at zero external momentum. It is convenient to take the trace of the matrices corresponding to the graphs, after which the integrands may be simplified by the use of the properties of the matrix $\mathbf{S}$. The traces of products of $\mathbf{S}$ matrices can be expressed in terms of relatively simple combinations of scalar products of the internal momenta, and Gram determinants, and it is the properties of the latter which simplify the calculation drastically. The simplest examples of such relations are

$$\begin{aligned} \mathrm{Tr}\,\mathbf{S}(k) &= d - 2 \\ \mathrm{Tr}\,\mathbf{S}(k)\mathbf{S}(p) &= d - 4\frac{\mathrm{Gr}\,(k, p)}{k^2p^2} \\ \mathrm{Tr}\,\mathbf{S}(k)\mathbf{S}(p)\mathbf{S}(q) &= d - 2 - 4\frac{\mathrm{Gr}\,(k, p, q)}{k^2p^2q^2} \end{aligned} \tag{9}$$

where $\mathrm{Gr}\,(\boldsymbol{k},\boldsymbol{p})$ and $\mathrm{Gr}\,(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})$ are the Gram determinants of two and three vectors, respectively

$$\mathrm{Gr}\,(\boldsymbol{k},\boldsymbol{p}) = \begin{vmatrix} \boldsymbol{k}^2 & \boldsymbol{kp} \\ \boldsymbol{pk} & \boldsymbol{p}^2 \end{vmatrix}$$
$$\mathrm{Gr}\,(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) = \begin{vmatrix} \boldsymbol{k}^2 & \boldsymbol{kp} & \boldsymbol{kq} \\ \boldsymbol{pk} & \boldsymbol{p}^2 & \boldsymbol{pq} \\ \boldsymbol{qk} & \boldsymbol{qp} & \boldsymbol{q}^2 \end{vmatrix} . \tag{10}$$

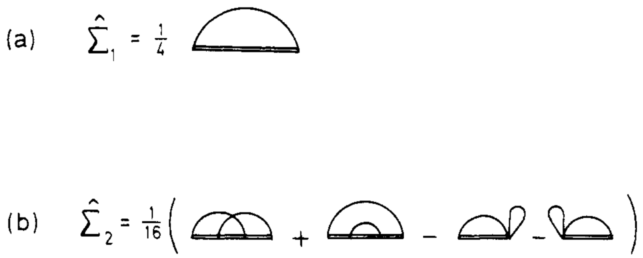

Figure 1. One-loop (a) and two-loop (b) graphs of the diagrammatic expression for the 'self-energy' matrix $\hat{\boldsymbol{\Sigma}}$ defined by equation (8). The ordinary lines correspond to the regularized correlation function (4) $C_{\mathrm{reg}}(\boldsymbol{p}) = g/(M^2 + \boldsymbol{p}^2)^{1+\alpha}$, and the double lines to the matrix $S_{mn}(\boldsymbol{p}) = \delta_{mn} - 2p_m p_n/p^2$. The coefficients from the expansion of the function $U = \tanh(\psi/2)$ together with the combinatorial factors are indicated explicitly, and thus all the vertex factors in the graphs are equal to unity.

For Σ_1 we readily obtain $\Sigma_1 = [(d-2)/4d(2\pi)^d] \int \mathrm{d}\boldsymbol{p}\, C_{\mathrm{reg}}(\boldsymbol{p})$, which yields $Z_1 = -\alpha g/(4\pi)^{1+\alpha}\Gamma(2+\alpha)\varepsilon$. Here, $C_{\mathrm{reg}}(\boldsymbol{p})$ is the regularized correlation function (4) in momentum space. In the S-matrix representation, the earlier two-loop result [8] may be verified without any calculation of graphs. Indeed, from the relations (9) it follows, that the sum of the four two-loop graphs of figure 1 is equal to an integral, which contains the three vector Gram determinant in the integrand (the trivial contributions proportional to the square of the one-loop graph cancel). However, the Gram determinat of linearly dependent vectors vahishes, therefore we immediately see that $\mathrm{Tr}\,\hat{\boldsymbol{\Sigma}}_2 = 0$, thus $\Sigma_2 = 0$, too.

For the same reason in the trace of the last eight graphs of figure 2 only the trivial parts proportional to $(d-2)[\int \mathrm{d}\boldsymbol{p}\, C_{\mathrm{reg}}(\boldsymbol{p})]^3$ survive. Due to this and the relation $\mathrm{Tr}\,\mathbf{S} = d-2$ the total contribution of the last 12 graphs of figure 2 to Σ_3 is $-7d^3\Sigma_1/3(d-2)^2$. For the graphs form the 11th to 14th parts of figure 2 with products of three $\mathbf{S}$ matrices the last relation in (9) yields expressions, in which the scalar products of different internal momenta enter only in the Gram determinant. The most difficult graphs are the first ten in figure 2, which involve products of five $\mathbf{S}$ matrices. The relations (9) are not sufficient to carry out the necessary algebraic transformations, and it is convenient to introduce the symmetric matrix $\mathbf{M}$

$$M_{mn}(\boldsymbol{p},\boldsymbol{q}) \equiv [\boldsymbol{p}^2\boldsymbol{q}^2 - (\boldsymbol{pq})^2]\delta_{mn} - p_m p_n \boldsymbol{q}^2 - q_m q_n \boldsymbol{p}^2 + (p_m q_n + p_n q_m)\boldsymbol{pq} \tag{11}$$

with the properties $\mathbf{M}(\boldsymbol{p},\boldsymbol{q})\boldsymbol{p} = 0$, $\boldsymbol{k}\mathbf{M}(\boldsymbol{p},\boldsymbol{q})\boldsymbol{k} = \mathrm{Gr}\,(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})$. To proceed, we need expressions for traces of products of $\mathbf{S}$ and $\mathbf{M}$ matrices, e.g.

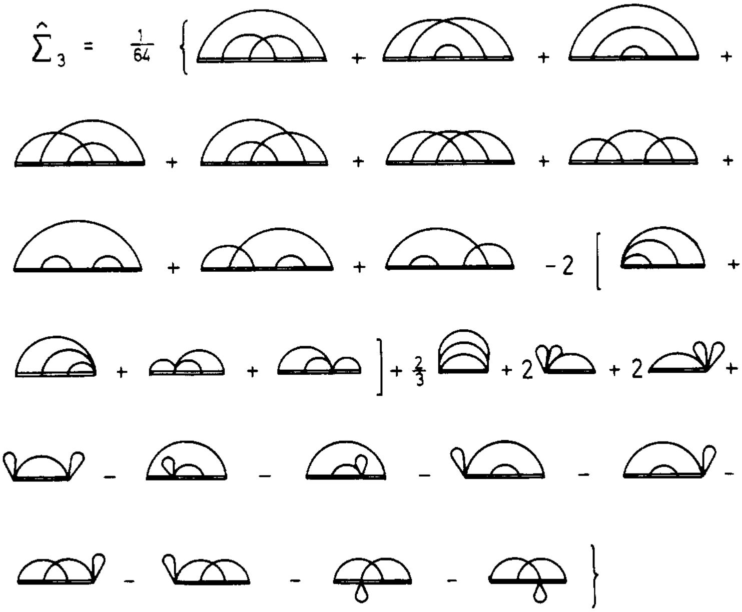

Figure 2. Three-loop graphs corresponding to $\hat{\Sigma}_3$. The graphical rules are the same as in figure 1.

$$\mathrm{Tr}\,\mathbf{S}(\boldsymbol{k})\mathbf{M}(\boldsymbol{p},\boldsymbol{q}) = (d-2)\mathrm{Gr}\,(\boldsymbol{p},\boldsymbol{q}) - 2\frac{\mathrm{Gr}\,(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})}{\boldsymbol{k}^2}$$

$$\mathrm{Tr}\,\mathbf{S}(\boldsymbol{p}+\boldsymbol{q})\mathbf{S}(\boldsymbol{q})\mathbf{M}(\boldsymbol{k},\boldsymbol{p}) = (d-2)\mathrm{Gr}\,(\boldsymbol{p},\boldsymbol{q}) - 2\frac{\boldsymbol{p}^2\mathrm{Gr}\,(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})}{\boldsymbol{q}^2(\boldsymbol{p}+\boldsymbol{q})^2}$$

$$\mathrm{Tr}\,\mathbf{M}(\boldsymbol{k},\boldsymbol{p})\mathbf{M}(\boldsymbol{k},\boldsymbol{p}+\boldsymbol{q}) = (d-2)\mathrm{Gr}\,(\boldsymbol{k},\boldsymbol{p})\mathrm{Gr}\,(\boldsymbol{k},\boldsymbol{p}+\boldsymbol{q}) - \boldsymbol{k}^2\mathrm{Gr}\,(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\,.$$

Using these and analogous relations we express the fourteen non-trivial graphs of figure 2 in the form

$$D_i = \frac{(d-2)}{d}\left[\iint \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^d} C_{\mathrm{reg}}(\boldsymbol{p})\right]^3 - \frac{4}{d}\int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^d}\int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^d}\int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^d} \times C_{\mathrm{reg}}(\boldsymbol{k})C_{\mathrm{reg}}(\boldsymbol{p})C_{\mathrm{reg}}(\boldsymbol{q})\mathrm{Gr}\,(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})I_i(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) \qquad (12)$$

where D_i denotes the full contribution to Σ_3 of the ith graph in figure 2, C_{reg} is the regularized correlation function of the random potential, and the functions I_i, $i = 1, 2, \ldots, 14$ are listed in the table 1. To proceed with the calculation of these integrals, we use the following lemma to remove the Gram determinant from the integrand.

Lemma. The formula of dimensional transformation

$$\int \mathrm{d}^d\boldsymbol{x}\int \mathrm{d}^d\boldsymbol{y}\int \mathrm{d}^d\boldsymbol{z}\,\mathrm{Gr}\,(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})f(a_1\boldsymbol{x}^2 + a_2\boldsymbol{y}^2 + a_3\boldsymbol{z}^2 + 2a_4\boldsymbol{x}\boldsymbol{y} + 2a_5\boldsymbol{x}\boldsymbol{z} + 2a_6\boldsymbol{y}\boldsymbol{z})$$
$$= \frac{(d-2)(d-1)d}{(2\pi)^3}\int \mathrm{d}^{d+2}\boldsymbol{x}\int \mathrm{d}^{d+2}\boldsymbol{y}\int \mathrm{d}^{d+2}\boldsymbol{z} \times f(a_1\boldsymbol{x}^2 + a_2\boldsymbol{y}^2 + a_3\boldsymbol{z}^2 + 2a_4\boldsymbol{x}\boldsymbol{y} + 2a_5\boldsymbol{x}\boldsymbol{z} + 2a_6\boldsymbol{y}\boldsymbol{z})$$

Table 1. Expressions for the functions $I_i(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})$, which determine the non-trivial parts of the integrals corresponding to the first fourteen graphs of figure 2 in the same order.

$I_1 = \dfrac{1}{(\boldsymbol{k}+\boldsymbol{p})^2(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})^2(\boldsymbol{k}+\boldsymbol{q})^2}$	$I_8 = \dfrac{1}{(\boldsymbol{k}+\boldsymbol{p})^2\boldsymbol{k}^2(\boldsymbol{k}+\boldsymbol{q})^2}$
$I_2 = \dfrac{1}{\boldsymbol{k}^2\boldsymbol{p}^2(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})^2}$	$I_9 = \dfrac{1}{\boldsymbol{k}^2(\boldsymbol{k}+\boldsymbol{p})^2(\boldsymbol{p}+\boldsymbol{q})^2}$
$I_3 = 0$	$I_{10} = \dfrac{1}{(\boldsymbol{k}+\boldsymbol{p})^2(\boldsymbol{k}+\boldsymbol{q})^2\boldsymbol{q}^2}$
$I_4 = \dfrac{[\boldsymbol{k}^2+2\boldsymbol{k}\boldsymbol{p}]^2}{\boldsymbol{k}^2(\boldsymbol{k}+\boldsymbol{p})^2(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})^2(\boldsymbol{p}+\boldsymbol{q})^2\boldsymbol{p}^2}$	$I_{11} = \dfrac{1}{(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})^2(\boldsymbol{p}+\boldsymbol{q})^2\boldsymbol{q}^2}$
$I_5 = \dfrac{[\boldsymbol{q}^2+2\boldsymbol{k}\boldsymbol{q}]^2}{\boldsymbol{k}^2(\boldsymbol{k}+\boldsymbol{p})^2(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})^2(\boldsymbol{k}+\boldsymbol{q})^2\boldsymbol{q}^2}$	$I_{12} = \dfrac{1}{\boldsymbol{k}^2(\boldsymbol{k}+\boldsymbol{p})^2(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})^2}$
$I_6 = \dfrac{[(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})^2-2\boldsymbol{k}\boldsymbol{q}]^2}{\boldsymbol{k}^2(\boldsymbol{k}+\boldsymbol{p})^2(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})^2(\boldsymbol{p}+\boldsymbol{q})^2\boldsymbol{q}^2}$	$I_{13} = \dfrac{1}{\boldsymbol{k}^2(\boldsymbol{p}+\boldsymbol{q})^2\boldsymbol{q}^2}$
$I_7 = \dfrac{\boldsymbol{p}^2}{\boldsymbol{k}^2(\boldsymbol{k}+\boldsymbol{p})^2(\boldsymbol{p}+\boldsymbol{q})^2\boldsymbol{q}^2}$	$I_{14} = \dfrac{1}{\boldsymbol{k}^2(\boldsymbol{k}+\boldsymbol{p})^2\boldsymbol{q}^2}$

holds for functions f such that the integrals involved converge.

The proof is given in appendix 1. Using Feynman parametrization and differential operators we may cast all the integrals of the type (12) in a form in which the momentum integrals are of the type required in the lemma. On the other hand, the parametric integrals do not depend on the dimensionality of the momentum space. Therefore, the lemma allows us to omit the Gram determinant in the integrals corresponding to the non-trivial parts of the first 14 graphs of figure 2 with simultaneous change in the dimensionality of the integrals by two and multiplication of the integrands by $(d-2)(d-1)d/(2\pi)^3$. The resulting graphs in the $(d+2)$-dimensional space are logarithmically divergent, and a great deal of them contain also divergent subgraphs. For conveniency of calculations, we rearranged the sum of the integrals to obtain simplest possible graphs with subdivergences, thus

$$\sum_{i=1}^{14} D_i = \frac{2(d-2)}{d}\left[\iint \frac{\mathrm{d}p}{(2\pi)^d} C_{\mathrm{reg}}(p)\right]^3 - \frac{4(d-2)(d-1)}{(2\pi)^{3(d+1)}} \times \int \mathrm{d}^{d+2}k \int \mathrm{d}^{d+2}p \int \mathrm{d}^{d+2}q C_{\mathrm{reg}}(k) C_{\mathrm{reg}}(p) C_{\mathrm{reg}}(q) \sum_{i=1}^{6} K_i(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})$$

with the expressions for the functions K_i, $i = 1,2,\ldots,6$ listed in table 2. Graphs corresponding to functions $K_1 - K_4$ do no contain subdivergences, whereas those corresponding to K_5 and K_6 do, but they have simpler structure, and therefore are calculable.

We need singular in ε contributions of the graphs only. For graphs without subdivergences no subtractions are necessary, and since the residue of the pole $1/\varepsilon$ in such a graph does not depend on dimensional parameters, we may choose the masses in all its lines nearly arbitrarily, the only restriction being that we must not introduce infrared divergences by the choice of masses. Thus, we set $M = 0$ in two of the three massive propagators of the graphs corresponding to K_1–K_4, obtaining an integral of

Table 2. Expressions for the functions $K_i(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})$, which correspond to actually calculated graphs. Graphs with K_1–K_4 are superficially divergent only, whereas those with K_5 and K_6 contain subdivergences.

$K_1 = -\dfrac{4(\boldsymbol{kp})(\boldsymbol{pq})}{\boldsymbol{k}^2\boldsymbol{p}^2(\boldsymbol{k}+\boldsymbol{p})^2(\boldsymbol{k}+\boldsymbol{q})^2(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})^2}$	$K_4 = \dfrac{2(\boldsymbol{kq})\boldsymbol{p}^2}{\boldsymbol{k}^2\boldsymbol{q}^2(\boldsymbol{k}+\boldsymbol{p})^2(\boldsymbol{p}+\boldsymbol{q})^2(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})^2}$
$K_2 = -\dfrac{2\boldsymbol{kp}}{\boldsymbol{k}^2\boldsymbol{p}^2(\boldsymbol{p}+\boldsymbol{q})^2(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})^2}$	$K_5 = \dfrac{1}{\boldsymbol{k}^2(\boldsymbol{k}+\boldsymbol{p})^2(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})^2}$
$K_3 = \dfrac{2\boldsymbol{kq}}{\boldsymbol{k}^2(\boldsymbol{k}+\boldsymbol{p})^2(\boldsymbol{p}+\boldsymbol{q})^2(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})^2}$	$K_6 = -\dfrac{1}{\boldsymbol{k}^2\boldsymbol{q}^2(\boldsymbol{k}+\boldsymbol{p})^2}$

a product of a massless finite two-loop graph of self-energy type and a massive propagator C_{reg}. A massless two-loop self energy graph is proportional to a power of the external momentum multiplied by a function of dimensionality and the exponents of the powerlike lines. The integral of C_{reg} multiplied by a power of momentum is readily calculable with the result

$$\int \mathrm{d}\boldsymbol{k}\frac{1}{(\boldsymbol{k}^2+M^2)^{\alpha}\boldsymbol{k}^{2\beta}} = \frac{\pi^{\mu}\Gamma(\alpha+\beta-\mu)\Gamma(\mu-\beta)}{\Gamma(\alpha)\Gamma(\mu)M^{2(\alpha+\beta-\mu)}}\,. \tag{13}$$

Here, and henceforth we use the convention [10] $\mu \equiv d/2$. The value of the exponent of the power function is found from dimensional analysis of the integral corresponding to the two-loop graph, and the task is to calculate the coefficient function, which we shall call the value of the graph.

For the graphs corresponding to K_1–K_4 the pole $1/\varepsilon$ arises from the integral with C_{reg}, whereas all the two-loop massless graphs turn out to be finite. To compute them we have used the method of uniqueness [10], which is the most convenient means for such a task. In fact, we did not need to use all the tools of this powerful method, but only the various recursion relations and the transformation group of the master two-loop graph. In this method the calculations are carried out in coordinate space. Instead of using the Fourier transformation we simply reorganize the lines on a momentum graph in such a way that the integration variables sit at vertices instead of flowing in loops. We present here the main stages of the calculation of the graph corresponding to K_3, which is relatively simple, but which contains all the essential ingredients of the method.

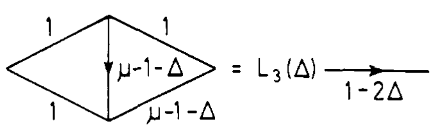

Figure 3. Two-loop massless graph from the calculation of the integral with K_3. The graph is computed in the coordinate space, in which each vertex carries a coordinate variable $\boldsymbol{x}$. The coordinates of the vertices on the left and on the right are fixed, whereas those in the middle of the graph are integrated out. A line with an index α between vertices with coordinates $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ corresponds to the function $1/(\boldsymbol{x}_1 - \boldsymbol{x}_2)^{2\alpha}$, and an arrow pointing from $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$ represents the vector $\boldsymbol{x}_2 - \boldsymbol{x}_1$. The coefficient function L_3 is called the value of the graph, and the quantity Δ has been introduced to regularize the intermediate divergences in the calculations. Eventually, only the value of the graph for $\Delta = 0$ is needed.

Performing the integral with the remaining massive line, we obtain the two-loop graph depicted in figure 3, which still contains a vector argument. To each vertex we prescribe a coordinate variable, a line between two vertices corresponds to a power function of the difference of the coordinates at the vertices according to the following rule: a line with the index a between vertices $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ corresponds to the function $1/(\boldsymbol{x}_1 - \boldsymbol{x}_2)^{2a}$. The graph is calculated for $\varepsilon = 0$, therefore the index $\mu = 2 + \alpha$. The arrow pointing from $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$ represents the vector $\boldsymbol{x}_2 - \boldsymbol{x}_1$. Finally, there is an integral over the variables of the two vertices in the middle of the graph, whereas the variables at both ends of the graph (as well as at the ends of the line on the right hand side of figure 3) are free parameters. In the graph of figure 3, we have introduced a regularising parameter Δ in view of the divergences which appear in the intermediate calculations (but which, of course, cancel in the final result, since the integral corresponding to the graph is convergent at $\Delta = 0$). To find the value of the graph, we multiply the relation of figure 3 by $\boldsymbol{x}$ (we use this variable to remind ourselves that we now regard the graph as a coordinate space graph), and expressing the scalar product on the left-hand side in the form of a sum of squares, we obtain for the coefficient function L_3 the relation shown in figure 4.

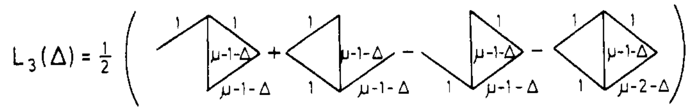

Figure 4. The recursion relation for the value L_3 of the graph of figure 3. The three first graphs on the right-hand side of this equation may be calculated by the chain rule (14), whereas the last one leads to the function h.

The three first graphs on the right-hand side of figure 4 may be calculated by the 'chain rule', which allows us graphically to replace two lines connected by an integration vertex by a single line; analytically it is the following formula for the convolution of two power functions:

$$\int \mathrm{d}\boldsymbol{y} \frac{1}{(\boldsymbol{x}-\boldsymbol{y})^{2\alpha}\boldsymbol{y}^{2\beta}} = \frac{\pi^{\mu}\Gamma(\mu-\alpha)\Gamma(\mu-\beta)\Gamma(\alpha+\beta-\mu)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(2\mu-\alpha-\beta)\boldsymbol{x}^{2(\alpha+\beta-\mu)}}. \tag{14}$$

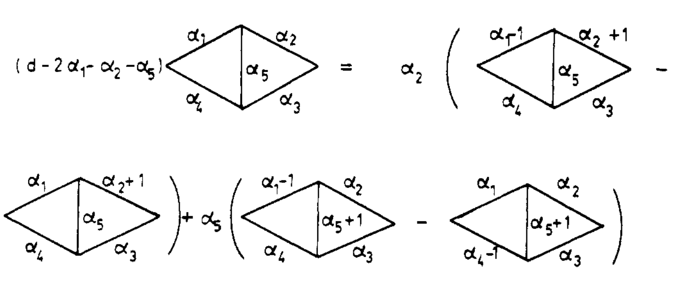

Figure 5. The recursion relation, which allows us to express the last graph of figure 4 as a sum of graphs calculable by the chain rule, and a graph corresponding to the function h.

To calculate the fourth graph, we use the recursion formula [10], shown graphically in figure 5, which allows us to reduce the graph to three new graphs calculable by the chain rule, and a graph which we were not able to calculate in a closed form. The value of this graph is proportional to the function h defined by the relation

$$h(\mu) \equiv \frac{\boldsymbol{x}^2}{\pi^{2\mu}} \int \mathrm{d}\boldsymbol{x}_1 \mathrm{d}\boldsymbol{x}_2 \frac{1}{\boldsymbol{x}_1^2 \boldsymbol{x}_2^2 (\boldsymbol{x}_1 - \boldsymbol{x}_2)^{2(\mu-1)} (\boldsymbol{x} - \boldsymbol{x}_2)^2 (\boldsymbol{x} - \boldsymbol{x}_1)^{2(\mu-1)}} \,. \tag{15}$$

Here, the dimension of space is $d = 2\mu$. However, we succeeded in calculating the values of the function h for even integer values of dimensionality. The calculation and the properties of the function h are presented in appendix 2. We would like to point out that with the aid of the transformation group of the master two-loop graph several other graphical representations of the function h may be derived; these are depicted in figure 6.

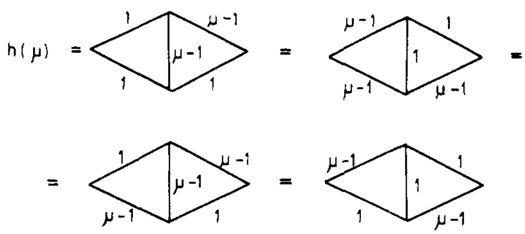

Figure 6. Different graphical expressions for the function h regarded as the value of the corresponding graph (i.e. the coefficient of the overall power function of the coordinate). These equations were obtained with the aid of the transformation group of the master two-loop graph [10].

The graphs corresponding to K_1, K_2 and K_4 were calculated in a similar fashion with the following result at the leading order of the ε expansion

$$\int \mathrm{d}^{d+2}\boldsymbol{k} \int \mathrm{d}^{d+2}\boldsymbol{p} \int \mathrm{d}^{d+2}\boldsymbol{q} C_{\mathrm{reg}}(\boldsymbol{k}) C_{\mathrm{reg}}(\boldsymbol{p}) C_{\mathrm{reg}}(\boldsymbol{q}) \sum_{i=1}^{4} K_i(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$$
$$= \frac{2\pi^{3(\alpha+2)}}{3\varepsilon\Gamma^3(2+\alpha)} \Big\{ \Gamma^2(2+\alpha)(\alpha^2 - 3\alpha + 1) h(2+\alpha) + (\alpha^2 - \alpha - 3)$$
$$\times [\psi'(1+\alpha) - \psi'(1)] + 2 - \frac{2}{(1+\alpha)^2} \Big\} + \mathrm{O}(1)$$

where ψ' is the trigamma function.

The graphs corresponding to K_5 and K_6 contain subdivergences, therefore we must be careful with the masses. For the sum

$$K_5 + K_6 = -\frac{(\boldsymbol{k}+\boldsymbol{p})^2 + 2\boldsymbol{k}\boldsymbol{q} + 2\boldsymbol{p}\boldsymbol{q}}{\boldsymbol{k}^2 \boldsymbol{q}^2 (\boldsymbol{k}+\boldsymbol{p})^2 (\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})^2}$$

the large-momentum behaviour of the corresponding integrand is such that replacing the factor $1/(\boldsymbol{p}^2 + M^2)^{1+\alpha}$ by $1/\boldsymbol{p}^{2(1+\alpha)}$ we change the value of the integral by a finite in ε quantity only, which is irrelevant from the point of view of the RG analysis. After this the integral with K_6 can be calculated using equation (13) and the chain rule (14). The integrand with K_5 becomes proportional to

$$\frac{1}{(\boldsymbol{k}^2 + M^2)^{1+\alpha}(\boldsymbol{q}^2 + M^2)^{1+\alpha}\boldsymbol{p}^{2(1+\alpha)}\boldsymbol{k}^2(\boldsymbol{k}+\boldsymbol{p})^2(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})^2}$$

and it is not difficult to see that the singular in ε terms of the corresponding integral are not affected by the substitution $1/(\boldsymbol{q}^2 + M^2)^{1+\alpha} \rightarrow 1/\boldsymbol{q}^{2(1+\alpha)}$ after which the relations (13) and (14) are again sufficient to compute the resulting integral.

4. Discussion

The calculations of the previous section yield the following three-loop contribution to the renormalization constant Z:

$$Z_3 = \frac{Z_1^3}{6} + \frac{\alpha(1+2\alpha)g^3}{48(4\pi)^{3+3\alpha}\Gamma^3(2+\alpha)\varepsilon}\Big\{\Gamma^2(2+\alpha)(\alpha^2 - 3\alpha + 1)h(2+\alpha) + (\alpha^2 - \alpha - 3)[\psi'(1+\alpha) - \psi'(1)] + 2\Big\}.$$

The function $h(\mu)$ is defined by equation (15), and its properties are analysed in the appendix 2. In particular, its values can be calculated for integer values of the argument greater than one (for $\mu \leq 1$ the integral, which defines the function h, diverges). For example, $h(2) = 6\zeta(3) \approx 7.212$ and $h(3) = \zeta(3) - 1/3 \approx 0.8687$, where ζ is the Riemann zeta function. In particular, we conclude from these results that the three-loop contribution to the anomalous dimension of the diffusivity γ_D does not vanish identically, contrary to earlier conjectures [6,5]. For γ_D we obtain

$$\gamma_D = \frac{g}{(4\pi)^{1+\alpha}\Gamma(2+\alpha)} + \frac{\alpha(1+2\alpha)g^3}{48(4\pi)^{3+3\alpha}\Gamma^3(2+\alpha)}\Big\{\Gamma^2(2+\alpha)(\alpha^2 - 3\alpha + 1)h(2+\alpha) + (\alpha^2 - \alpha - 3)[\psi'(1+\alpha) - \psi'(1)] + 2\Big\}. \qquad (16)$$

The three-loop contribution vanishes with α as it should, and it also vanishes at $\alpha = -1/2$, which corresponds to $d_c = 1$. Using the exact solution of the one-dimensional stationary problem, equations (1) and (2) it has been shown [7] that the expression (16) is indeed valid also in the one-dimensional case, and, moreover, that the total anomalous dimension γ_D is exactly equal to its one-loop value, as in the two-dimensional case.

In conclusion, we have shown by explicit calculation that, contrary to earlier conjectures, the three-loop contribution to the anomalous dimension of the diffusion coefficient for the model of diffusion in a potential random field does not vanish identically, but only in the one- and two dimensional cases, in which the one-loop contribution yields the exact value of the anomalous dimension. We have explained in detail the method of calculation, which includes novel features such as the use of Gram determinants and dimensional transformation to simplify the structure of integrals corresponding to multi-loop graphs with vector propagators. These tricks also allow us to derive new useful recursion relations for functions defined by massless Feynman graphs.

Appendix 1.

In this appendix we give the proof of the lemma of dimensional transformation. In the integral

$$J \equiv \int \mathrm{d}^d\boldsymbol{x} \int \mathrm{d}^d\boldsymbol{y} \int \mathrm{d}^d\boldsymbol{z}\, \mathrm{Gr}\,(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) f(a_1\boldsymbol{x}^2 + a_2\boldsymbol{y}^2 + a_3\boldsymbol{z}^2 + 2a_4\boldsymbol{x}\boldsymbol{y} + 2a_5\boldsymbol{x}\boldsymbol{z} + 2a_6\boldsymbol{y}\boldsymbol{z})$$

we change variables to express the argument of the function f in the form of a sum of squares of the integration variables. The corresponding Jacobi determinant is equal to unity, and since the Gram determinant does not change under such a transformation, we obtain

$$J = \int \mathrm{d}^d\boldsymbol{x} \int \mathrm{d}^d\boldsymbol{y} \int \mathrm{d}^d\boldsymbol{z}\, \mathrm{Gr}\,(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) f(b_1\boldsymbol{x}^2 + b_2\boldsymbol{y}^2 + b_3\boldsymbol{z}^2)$$

where

$$b_1 = a_1$$

$$b_2 = a_2 - \frac{a_4^2}{a_1}$$

$$b_3 = a_3 - \frac{(a_1a_6 - a_4a_5)^2}{a_1(a_1a_2 - a_4^2)}\,.$$

Using the relation $\mathrm{Gr}\,(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = \boldsymbol{x}\mathbf{M}(\boldsymbol{y},\boldsymbol{z})\boldsymbol{x}$, where the matrix $\mathbf{M}$ is defined by equation (11), we may write

$$J = \int \mathrm{d}^d\boldsymbol{y} \int \mathrm{d}^d\boldsymbol{z}\, M_{mn}(\boldsymbol{y},\boldsymbol{z}) \int \mathrm{d}^d\boldsymbol{x}\, x_m x_n f(b_1\boldsymbol{x}^2 + b_2\boldsymbol{y}^2 + b_3\boldsymbol{z}^2)$$

where, due to rotational symmetry, the integral over $\boldsymbol{x}$ yields

$$\int \mathrm{d}^d\boldsymbol{x}\, x_m x_n f(b_1\boldsymbol{x}^2 + b_2\boldsymbol{y}^2 + b_3\boldsymbol{z}^2) = (\delta_{mn}/d) \int \mathrm{d}^d\boldsymbol{x}\, \boldsymbol{x}^2 f(b_1\boldsymbol{x}^2 + b_2\boldsymbol{y}^2 + b_3\boldsymbol{z}^2)$$
$$\equiv (\delta_{mn}/d) F(b_2\boldsymbol{y}^2 + b_3\boldsymbol{z}^2).$$

Since the trace of the matrix $\mathbf{M}$ is equal to $(d-2)[\boldsymbol{y}^2\boldsymbol{z}^2 - (\boldsymbol{y}\boldsymbol{z})^2]$, we arrive at the relation

$$J = \frac{d-2}{d} \int \mathrm{d}^d\boldsymbol{y} \int \mathrm{d}^d\boldsymbol{z}\, \mathrm{Gr}\,(\boldsymbol{y},\boldsymbol{z}) F(b_2\boldsymbol{y}^2 + b_3\boldsymbol{z}^2)$$
$$= \frac{d-2}{d} \int \mathrm{d}^d\boldsymbol{z}\, (\boldsymbol{z}^2\delta_{mn} - z_m z_n) \int \mathrm{d}^d\boldsymbol{y}\, y_m y_n F(b_2\boldsymbol{y}^2 + b_3\boldsymbol{z}^2)$$

in which the integral over $\boldsymbol{y}$ yields δ_{mn} multiplied by a scalar function of $\boldsymbol{z}^2$. Therefore, we conclude that

$$J = \frac{(d-2)(d-1)}{d^2} \int \mathrm{d}^d\boldsymbol{x} \int \mathrm{d}^d\boldsymbol{y} \int \mathrm{d}^d\boldsymbol{z}\, \boldsymbol{x}^2\boldsymbol{y}^2\boldsymbol{z}^2 f(b_1\boldsymbol{x}^2 + b_2\boldsymbol{y}^2 + b_3\boldsymbol{z}^2)\,.$$

In spherical coordinates, the angular integrals factorize, and the squares of the integration variables may be included in the radial part of the integration measure, which results in the shift $d \to d+2$ of the effective dimensionality. Introducing angular integrals corresponding to the higher dimensionality, we obtain

$$J = \frac{(d-2)(d-1)}{d^2}\left(\frac{\pi^d \Gamma(d/2+1)}{\pi^{d+2}\Gamma(d/2)}\right)^3 \times \int \mathrm{d}^{d+2}\boldsymbol{x} \int \mathrm{d}^{d+2}\boldsymbol{y} \int \mathrm{d}^{d+2}\boldsymbol{z}\, f(b_1\boldsymbol{x}^2 + b_2\boldsymbol{y}^2 + b_3\boldsymbol{z}^2)\,.$$

Inverting the transformation which led us to the sum of squares in the argument of the function f, we finally arrive at the conjecture of the lemma

$$\int \mathrm{d}^d\boldsymbol{x} \int \mathrm{d}^d\boldsymbol{y} \int \mathrm{d}^d\boldsymbol{z}\, \mathrm{Gr}\,(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) f(a_1\boldsymbol{x}^2 + a_2\boldsymbol{y}^2 + a_3\boldsymbol{z}^2 + 2a_4\boldsymbol{x}\boldsymbol{y} + 2a_5\boldsymbol{x}\boldsymbol{z} + 2a_6\boldsymbol{y}\boldsymbol{z})$$
$$= \frac{(d-2)(d-1)d}{(2\pi)^3} \int \mathrm{d}^{d+2}\boldsymbol{x} \int \mathrm{d}^{d+2}\boldsymbol{y} \int \mathrm{d}^{d+2}\boldsymbol{z}\, f(a_1\boldsymbol{x}^2 + a_2\boldsymbol{y}^2 + a_3\boldsymbol{z}^2 + 2a_4\boldsymbol{x}\boldsymbol{y} + 2a_5\boldsymbol{x}\boldsymbol{z} + 2a_6\boldsymbol{y}\boldsymbol{z})\,.$$

Appendix 2.

We were not able to calculate the function h defined by

$$h(\mu) \equiv \frac{\boldsymbol{x}^2}{\pi^{2\mu}} \int \mathrm{d}\boldsymbol{x}_1 \mathrm{d}\boldsymbol{x}_2\, \frac{1}{\boldsymbol{x}_1^2\boldsymbol{x}_2^2(\boldsymbol{x}_1-\boldsymbol{x}_2)^{2(\mu-1)}(\boldsymbol{x}-\boldsymbol{x}_2)^2(\boldsymbol{x}-\boldsymbol{x}_1)^{2(\mu-1)}}$$

in a closed form, i.e. in the form of a finite combination of the Γ function and its derivatives (or some other well known special functions). The best we achieved was the derivation of a recursion formula for this function, and the 'initial condition' for this relation, which together allow to compute the values of the function h for positive integer values of the argument.

We obtained this recursion relation in a fashion similar to the derivation of the dimensional transformation lemma. Consider the integral

$$h(\mu+1) = \frac{\boldsymbol{x}^2}{\pi^{2(\mu+1)}} \int \mathrm{d}^{2\mu+2}\boldsymbol{x}_1 \mathrm{d}^{2\mu+2}\boldsymbol{x}_2\, \frac{1}{\boldsymbol{x}_1^2\boldsymbol{x}_2^2(\boldsymbol{x}_1-\boldsymbol{x}_2)^{2\mu}(\boldsymbol{x}-\boldsymbol{x}_2)^2(\boldsymbol{x}-\boldsymbol{x}_1)^{2\mu}}\,.$$

Inverting the procedure used to prove the lemma, we may write

$$h(\mu+1) = \frac{2}{(\mu-1)(2\mu-1)\pi^{2\mu}} \int \mathrm{d}^{2\mu}\boldsymbol{x}_1 \mathrm{d}^{2\mu}\boldsymbol{x}_2\, \frac{\mathrm{Gr}\,(\boldsymbol{x},\boldsymbol{x}_1,\boldsymbol{x}_2)}{\boldsymbol{x}_1^2\boldsymbol{x}_2^2(\boldsymbol{x}_1-\boldsymbol{x}_2)^{2\mu}(\boldsymbol{x}-\boldsymbol{x}_2)^2(\boldsymbol{x}-\boldsymbol{x}_1)^{2\mu}}$$

and expressing the Gram determinant as a combination of scalar products of the vectors $\boldsymbol{x}$, $\boldsymbol{x}_1$, and $\boldsymbol{x}_2$, we obtain a sum of graphs, which can be calculated by the chain rule, and a graph which yields $h(\mu)$. The result is

$$2(2\mu-1)(\mu-1)h(\mu+1) = \frac{1}{\Gamma^2(\mu)}\left\{3[\psi'(1)-\psi'(\mu)] - \frac{5}{(\mu-1)^2}\right\} + h(\mu)\,.$$

The initial value for the recursion we obtain for $h(2)$ from the expression for the function $\mathrm{ChT}(\alpha,\beta)$ defined by the master two-loop graph (depicted on the left-hand side of the equation in figure 5) with the following values of the exponents: $\alpha_1 = \alpha$, $\alpha_4 = \beta$, $\alpha_2 = \alpha_3 = \alpha_5 = \mu - 1 = \frac{1}{2}d - 1$. This graph has been calculated by both the Gegenbauer polynomial x-space technique [11], and the method of uniqueness [10]. Obviously, $h(2)$ is proportional to $\mathrm{ChT}(1,1)$ in four dimensions, therefore $h(2) = 6\zeta(3)$.

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